Estimating Poverty and Inequality from Grouped Data: How Well Do Parametric Methods Perform?

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Poverty and inequality are often estimated from grouped data as complete household surveys are neither always available to researchers nor easy to analyze. In this study we assess the performance of functional forms proposed by Kakwani (1980a) and Villasenor and Arnold (1989) to estimate the Lorenz curve from grouped data. The methods are implemented using the computational tools POVCAL and SimSIP, developed and distributed by the World Bank. To identify biases associated with these methods, we use unit data from several household surveys and theoretical distributions. We find that poverty and inequality are better estimated when the true distribution is unimodal than multimodal. For unimodal distributions, biases associated with poverty measures are rarely larger than one percentage point. For data from multi-peaked or heavily skewed distributions, the biases are likely to be higher and of unknown sign.

Keywords: grouped data, Lorenz curve, poverty, inequality, income distribution, POVCAL, SimSIP
JEL Classifications: C13, C14, C15, C16, D31, D63, I32

Introduction

Poverty and inequality are often estimated from grouped data (i.e., mean incomes of population quantiles, such as quintiles or deciles)[2] for two reasons: i) complete household surveys are not always available to researchers, and ii) the analysis of unit data is often labour- and time-intensive. Consequently, estimates of regional and global poverty and inequality are often derived from grouped data (Yotopoulos, 1989; Bourrguignon and Morrisson, 2002; Chen and Ravallion, 2001; 2004; Sala-i-Martin, 2006; and Ackland, Dowrick, and Freyens, 2008). In particular, data in summary form has been the sole source of information on income distributions

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of large countries, such as China, which greatly influence the extent and trend of global poverty (Reddy and Minoiu, 2007).[3] Data on the size distribution of firms is also often published by governments only in grouped form (Golan, Judge, and Perloff, 1996), as is much historical data (Shorrocks and Wan, 2008).

In applied work, poverty and inequality are estimated from grouped data almost exclusively by using POVCAL and SimSIP, software programs developed and distributed by the World Bank.[4] The two computational tools are essentially identical, except that SimSIP provides an Excel-based user-friendly interface, while POVCAL operates in MS-DOS. Both programs have been used extensively in poverty analysis (e.g., Belkacem and Limam, 2004; Bhallà, 2002; Chen and Ravallion, 2001; 2004; 2006; Chen and Wang, 2001; Figini and Santarelli, 2006; karshenas, 2004; Pritchett, 2006; Son and Kakwani, 2006; and Woo, Shi, Ximing, Xiaoying, and Xingpeng, 2004). POVCAL-based Gini coefficients have been included in the databases of the United Nations University, World Institute for Development Economics Research UNU-WIDER, 2005 and the World Bank's Measuring Income Inequality (Deininger and Squire, 1996). A large number of cross-country econometric analyses have subsequently been undertaken using these databases (e.g., Lundberg and Squire, 2003; Banerjee and Duflo, 2003; Milanovic, 2002; Forbes, 2000; Easterly, 1999; and Deininger and Squire, 1998). Finally, POVCAL and SimSIP have been widely used in the preparation of national poverty assessments (for instance, Ali and Elbadawi, 2008; Asra, 2000; Belkacem, 2001; Eele, Semboja, Likwelile, and Ackroyd, 2000; Acharya, 2004; and Joekes, Ahmed, Ercelawn, and Zaidi, 2000) and have been widely recommended to practitioners by development agencies (USAID, 2004; and World Bank, 2003).

Despite their widespread use, few studies have provided a systematic analysis of the grouped data methods used by POVCAL and SimSIP. This paper attempts to fill this gap. We analyze the accuracy of estimates based on two Lorenz curve functional forms used in the programs. These functional forms were developed respectively by Villasenor and Arnold (1989) and Kakwani (1980a), and are also known as the Generalized Quadratic (GQ) and Beta parameterizations. We compare estimates of Lorenz curves, poverty, and inequality derived from grouped data with those estimated directly from unit data, for a wide range of income distributions. We undertake both Monte Carlo simulations as well as deterministic comparisons. Our data sources comprise: household surveys from five countries (Brazil, China, Nicaragua, Tanzania, and Vietnam) and data generated from two distributions (the theoretical Dagum distribution and an empirical global income distribution).

We find that the two functional forms perform relatively well in estimating poverty from unimodal distributions. Larger biases were identified in the case of multi-peaked distributions. We also find that the biases vary (albeit not systematically) with the sample size, functional forms, distributions, poverty lines, and poverty indicators. Inequality (measured by the Gini coefficient) is accurately estimated in most cases considered. While we have attempted to analyze a wide range of possible income distributions, we caution that our results should be regarded as conditional on the data used and may therefore not hold for data derived from a different underlying distribution (or data generating process).

The remainder of the paper is structured as follows. The next section briefly discusses previous assessments of various techniques for parametrically estimating Lorenz curves from grouped data. In the following section, we present the biases identified based on Monte Carlo simulations. A next section presents our findings in the case of deterministic comparisons using household survey data. Conclusions are drawn in a final section. Plots of all distributions used in this article are presented in Figures A1 and A2 in the Appendix.

Previous studies

A limited literature examines the theoretical validity and empirical performance of alternative Lorenz curve functional forms. Villasenor and Arnold (1989) used the 1967-68 Australian Survey of Consumer Expenditure to find that GQ is superior to three alternative parameterizations (Kakwani and Podder, 1976; Pakes, 1981; and the classical Pareto distribution) based on the sum of squared errors over the entire support. Larger samples were associated with a better fit. Although the best fit of the Lorenz curve was achieved for unimodal distributions, the authors judged their parameterization to be satisfactory for bimodal income distributions as well. Kakwani (1980a) assessesed the goodness-of-fit of the Beta functional form using the 1974 Australian Household Expenditure Survey. The R-squared statistics from the estimating regressions (not reported in this article) were close to 0.99, while the fitted Lorenz curve values were within two decimal places of the survey values.

Ravallion and Huppi (1989) used household consumption data for 50,000 Indonesian households and compared Lorenz curve estimates obtained from three functional forms: Villasenor and Arnold (1989), Kakwani and Podder (1976), and Kakwani (1980a). They found that the worst fit was provided by the two-parameter specification of Kakwani and Podder (1976), while the other two specifications - also the subject of our study - gave broadly similar results.[5] The GQ parameterization provided a better fit towards the high end of the income distribution, while the Beta form did better in the left tail.

In relation to inequality, Cheong (2002) undertook a comparison of four Lorenz curve functional forms (Kakwani and Podder, 1976; Rasche, Gaffney, Koo, and Obst, 2000; Kakwani, 1980a; and Ortega, Martin, Fernandez, Ladoux, and Garcia, 1991) in estimating the US income Gini (for one hundred income classes). The author concluded that Kakwani's Beta form, although theoretically invalid (on which point see Anand, 1983; Anand and Kanbur, 1993a;b; Rasche, Gaffney, Koo, and Obst, 2000; and Ortega, Martin, Fernandez, Ladoux, and Garcia, 1991), provided

as good a fit to the data as did the theoretically valid parameterization proposed by Rasche, Gaffney, Koo, and Obst (2000). In an earlier study, Schader and Schmid (1994) used household survey data to compare Gini coefficients obtained through parametric Lorenz curve estimation with nonparametric Gastwirth bounds for the true Gini coefficient (Gastwirth, 1972). Estimates of inequality based on the Kakwani (1980a) parameterization of the Lorenz curve were found to lie between the nonparametric Gastwirth bounds for all datasets considered.

While many Lorenz curve functional forms have been proposed in the literature (other examples include Gastwirth and Glauberman, 1976; Gupta, 1984; Mazzarino, 1986; Basmann, Hayes, Slottje, and Johnson, 1990; and Ogwang and Rao, 1996), an exhaustive assessment of all the alternatives is not the object of this study. The main reason for restricting our attention to the GQ and Beta forms is their almost exclusive use in the estimation of poverty and inequality from grouped data, due to their use in the World Bank's POVCAL and SimSIP programs.

Findings from Monte Carlo simulations

In this section, we describe our Monte Carlo analysis of the performance of the GQ and Beta Lorenz curve parameterizations for grouped data. Our data are drawn first from a theoretical distribution - the Dagum - and second from a notional multimodal distribution.

Bandourian, Mcdonald, and Turley (2003) show that the Dagum distribution is the best-fitting among three parameter distributions to survey income data. Jenkins and Cox (1999) note that the Dagum provides a good fit to income distributions largely due to its ability to model skewed distributions. We parameterize it with median values from the reported best-fitting Dagum parameters for recent income distributions from 27 countries (Bandourian, Mcdonald, and Turley, 2003). These median parameter values happen to be closest to those fit by the authors for Russia's 1992 distribution.[6]

The multimodal distribution is the 2004 population-weighted world distribution of income, in which every individual is assigned the per capita Purchasing Power Parity-adjusted Gross Domestic Product (GDP) of her country (as reported in the World Development Indicators, 2006). The two higher peaks of the distribution correspond to a population mass concentrated at China's and India's per capita GDP, while the lower peak corresponds to the population mass of the rich nations (Figure 1).

From each hypothesized density, we draw 100 random samples of 1,000 observations.[7] We restrict the number of draws to 100 due to the high volume of manual work involved in running the software on these samples. Each sample is then collapsed into grouped data (quintile, decile, and ventile means) and entered into POVCAL. We do not consider cases beyond ventiles, since in practice at most twenty datapoints are typically available to researchers. Furthemore, we use multiple poverty lines in order to identify biases at multiple points along the support of the curve, including money-metric international poverty lines ($1/day and $2/day), nutritionally-anchored poverty lines (constructed by Reddy, Visaria, and Asali, 2008), and thresholds representing the population or survey median multiplied by various constant proportions.

Results for the Lorenz curve (Dagum distribution)

We first compare the Lorenz curve estimates from grouped data with the true curve. The goodness-of-fit is assessed both along the entire curve and in the left tail (up to a poverty headcount ratio of 20 per cent), using the Sum of Squared Errors (SSE) and the Sum of Absolute Errors (SAE). We find that a higher number of quantile means implies a lower SSE and SAE (for the GQ parameterization) but (surprisingly) the reverse is the case for the Beta parameterization (Table 1). If the SSE and SAE are computed up to a headcount ratio of 20 per cent, the Beta parameterization underperforms the GQ method, except in the case of quintile data.

Table 1
Sum of Squared Errors (SSE) and Sum of Absolute Errors (SAE) of the Lorenz curve estimate from grouped data

	GQ			Beta		
	Quintiles	Deciles	Ventiles	Quintiles	Deciles	Ventiles
Along the entire support						
SSE	0.3751	0.3706	0.3689	0.3637	0.3638	0.3620
SAE	5.3502	5.3165	5.3007	5.2458	5.2697	5.2704
Up to the 20th percentile of the population						
SSE	0.0021	0.0020	0.0020	0.0020	0.0024	0.0027
SAE	0.1622	0.1564	0.1526	0.1537	0.1707	0.1818

Average Lorenz curve estimates for population proportions up to 10 per cent (Table 2) give a fine-grained indication of the biases in the left tail of the distribution. A series of interesting patterns arise: for example, the GQ parametrization overestimates the income share accruing to each population proportion toward the left side of the support. In contrast, the Beta parameterization yields negative, and hence invalid, average Lorenz curve estimates for the bottom population centiles. POVCAL alerts the user to the invalidity of the Lorenz curve each time negative income shares are fitted.

The Lorenz curve is consistently overestimated across the support (Table 3). As a result, the true Lorenz curve is dominated by the estimated one (Figure 1), which

Table 2
Lorenz curve estimates from grouped data in the left tail of the distribution

Cum. pop. prop.	True	GQ Quintiles	GQ Deciles	GQ Ventiles	Beta Quintiles	Beta Deciles	Beta Ventiles
0.01	0.0001	0.0001	0.0000	0.0000	**-0.0016**	**-0.0010**	**-0.0007**
0.02	0.0002	0.0005	0.0004	0.0003	**-0.0017**	**-0.0008**	**-0.0004**
0.03	0.0005	0.0012	0.0010	0.0008	**-0.0011**	**-0.0001**	0.0005
0.04	0.0009	0.0021	0.0018	0.0017	**-0.0002**	0.0009	0.0016
0.05	0.0014	0.0032	0.0029	0.0028	0.0011	0.0023	0.0030
0.06	0.0020	0.0046	0.0043	0.0041	0.0026	0.0039	0.0047
0.07	0.0028	0.0062	0.0059	0.0056	0.0044	0.0058	0.0066
0.08	0.0037	0.0080	0.0077	0.0074	0.0064	0.0078	0.0086
0.09	0.0047	0.0101	0.0097	0.0094	0.0087	0.0101	0.0109

Note: Invalid Lorenz curve estimates are printed in boldface.

implies that distortions in the Lorenz curve arise along the entire support and any Lorenz-consistent measure of inequality derived from these estimates would underestimate inequality. The magnitude of the biases is very similar across sample sizes and estimation methods.

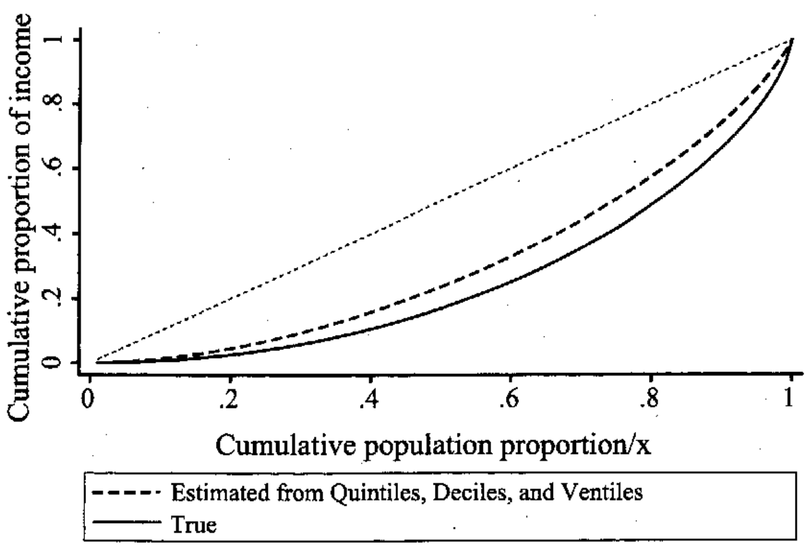
Table 3
Extent of Lorenz curve misestimation at along the entire support

Cum. pop.	GQ Quintiles	GQ Deciles	GQ Ventiles	Beta Quintiles	Beta Deciles	Beta Ventiles
0.1	111%	104%	100%	90%	115%	129%
0.2	81%	80%	80%	82%	86%	89%
0.3	63%	63%	63%	63%	64%	64%
0.4	49%	50%	50%	49%	49%	49%
0.5	39%	39%	40%	39%	39%	38%
0.6	31%	31%	31%	31%	31%	30%
0.7	24%	24%	24%	24%	24%	24%
0.8	18%	18%	18%	18%	18%	18%
0.9	12%	12%	12%	11%	12%	12%

Note: Positive values represent overestimates and negative values represent underestimates.

We repeated the exercise for the notional multimodal distribution. Using the SAE criterion, the GQ parametrization provides a worse fit. Interestingly, in larger samples, the goodness-of-fit does not vary monotonically with the number of datapoints for the Beta parameterization (Table 4). Furthermore, both functional forms occasionally give rise to invalid estimated Lorenz curves (Table 5). Finally, both methods underestimate the Lorenz curve (unlike in the case of the Dagum distribu-

Figure 1
True and fitted Lorenz curve, Dagum distribution
Lorenz curve based on GQ parameterization
Dagum distribution



Note: The fitted Lorenz curves from quintiles, deciles, and ventiles are observationally equivalent. The graph is identical for the Beta parameterization and is not shown.

tion) (Table 6).

Figure 2 offers a visual representation of these findings by superimposing the true and fitted Lorenz curves. Since both positive and negative biases occur along the support, the estimated and true Lorenz curves cross. It follows that whether a Lorenz-consistent inequality index will over- or underestimate the true level of inequality from these estimates depends on specific features of the index.

Results for poverty and inequality

We find that poverty and inequality are better estimated for the Dagum distribution than for the multimodal distribution (Tables 7-9). For the Dagum distribution, the two parameterizations perform exceptionally well. Specifically, the average bias associated with various poverty indicators is rarely higher than one percentage point (Table 7). Furthermore, the GQ parameterization slightly outperforms the Beta parameterization. As before, a larger sample is not always associated with a lower

Table 4
Sum of Squared Errors (SSE) and Sum of Absolute Errors (SAE) of the Lorenz curve estimate from grouped data

	GQ			Beta		
	Quintiles	Deciles	Ventiles	Quintiles	Deciles	Ventiles
Along the entire support						
SSE	0.5951	0.4558	0.5048	0.2506	0.0663	0.0620
SAE	2.8596	3.4789	3.6097	1.8910	1.5521	1.7965
Up to the 20th percentile of the population						
SSE	0.0000	0.0149	0.0141	0.0000	0.0001	0.0002
SAE	0.0194	0.5435	0.5300	0.0157	0.0389	0.0536

Table 5
Lorenz curve estimates from grouped data in the left tail of the distribtuion

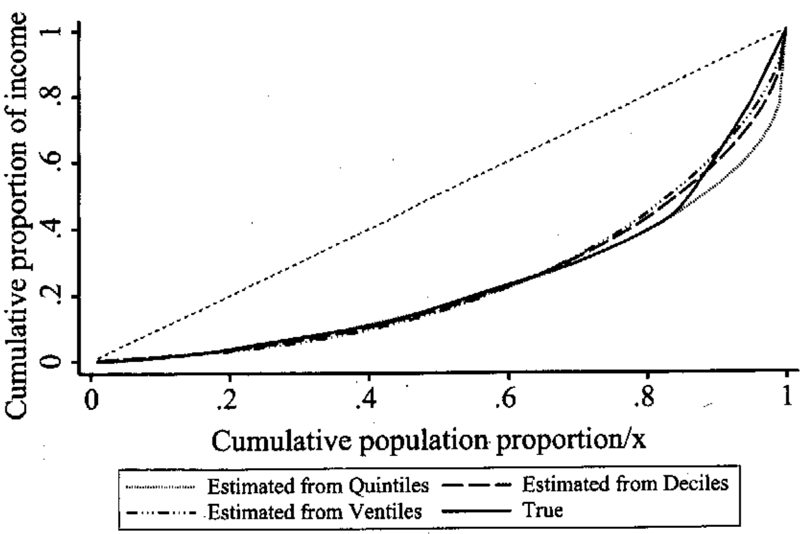
Cum. pop. prop.	True	GQ			Beta		
		Quintiles	Deciles	Ventiles	Quintiles	Deciles	Ventiles
0.01	0.0007	0.0005	**-0.0291**	**-0.0260**	**-0.0007**	0.0017	0.0023
0.02	0.0015	0.0012	**-0.0288**	**-0.0261**	**-0.0003**	0.0031	0.0040
0.03	0.0023	0.0020	**-0.0281**	**-0.0257**	0.0005	0.0045	0.0054
0.04	0.0033	0.0029	**-0.0270**	**-0.0250**	0.0015	0.0059	0.0067
0.05	0.0044	0.0040	**-0.0257**	**-0.0240**	0.0028	0.0072	0.0080
0.06	0.0057	0.0052	**-0.0241**	**-0.0228**	0.0042	0.0086	0.0092
0.07	0.0070	0.0065	**-0.0224**	**-0.0213**	0.0058	0.0101	0.0105
0.08	0.0083	0.0080	**-0.0204**	**-0.0196**	0.0075	0.0116	0.0117
0.09	0.0099	0.0095	**-0.0183**	**-0.0177**	0.0094	0.0131	0.0130

Table 6
Extent of Lorenz curve misestimation along the entire support

Cum. Pop. proportion	GQ Quintiles	GQ Deciles	GQ Ventiles	Beta Quintiles	Beta Deciles	Beta Ventiles
0.1	-5%	-235%	-231%	-4%	24%	21%
0.2	-7%	-61%	-63%	-2%	-7%	-16%
0.3	-6%	-22%	-23%	-3%	-12%	-20%
0.4	2%	-1%	-1%	3%	-5%	-12%
0.5	-1%	2%	2%	-1%	-5%	-9%
0.6	-1%	3%	3%	-3%	-3%	-4%
0.7	1%	6%	7%	0%	5%	6%
0.8	0%	5%	5%	0%	9%	13%
0.9	-18%	-14%	-14%	-15%	-3%	1%

Note: Positive values represent overestimates and negative values represent underestimates.

Figure 2
True and fitted Lorenz curve, multi-modal distribution
Lorenz curve based on Beta parameterization
Multi-modal distribution



Note: The graph corresponding to the GQ gives rise to the same qualitative conclusions and is not shown.

bias.

In contrast, the biases are often sizable for the multimodal distribution and larger samples are frequently associated with larger biases (Table 8).[8] In addition, the sign of the biases changes from positive (for quintile means) to negative (for decile and ventile means), as the poverty line rises. The Gini coefficient is more accurately estimated by the Beta parameterization, regardless of the distribution (Table 9). Biases are extremely small for the Dagum distribution, but as large as 5 percentage points (Dagum) and 3.7 percentage points (multimodal) in samples of quintiles.

From these results, it is difficult to identify any regularities other than that the biases are larger when the true distribution is multimodal. Indeed, Villasenor and Arnold (1989) themselves note that the GQ parameterization provides a good fit to data from the unimodal family of densities but less so to data with bimodal histograms.

Table 7
Poverty biases (in percentage points), Dagum distribution

Poverty indicator	Poverty line Median x:	True	GQ Quintiles	GQ Deciles	GQ Ventiles	Beta Quintiles	Beta Deciles	Beta Ventiles
Poverty headcount ratio	1.33	66.9	0.18	0.34	0.53	-0.04	-0.21	-0.21
	0.5	21.7	-0.49	-0.67	-0.76	-0.83	-0.48	-0.21
	0.33	13.0	-0.10	-0.25	-0.27	-1.07	-1.07	-0.98
	0.25	9.1	0.01	0.03	0.05	-1.07	-1.13	-1.18
Poverty gap ratio	1.33	31.7	0.09	0.05	0.10	0.08	0.09	0.19
	0.5	9.6	0.03	0.06	0.45	-0.03	-0.28	-0.30
	0.33	5.8	0.27	0.36	0.62	0.52	0.00	-0.22
	0.25	4.0	0.35	0.51	0.67	1.03	0.41	0.04
Squared poverty gap	1.33	19.9	0.07	0.04	0.07	0.19	0.07	0.09
	0.5	5.9	0.23	0.35	0.48	1.07	0.37	0.01
	0.33	3.5	0.39	0.55	0.68	2.20	1.16	0.54
	0.25	2.4	0.47	0.68	0.84	3.36	1.80	1.02

Findings from household surveys - Deterministic comparison results for poverty and inequality

In this section we report on the performance of the two functional forms using unit data from five household surveys (Tables 10-12).[9] We use disposable income and consumption distributions from nationally representative surveys of Brazil, China, Tanzania, Nicaragua, and Vietnam. A description of the data sources and variables is provided in the Appendix.

For many surveys, the biases in poverty estimates are very small: they are seldom larger than one percentage point (in either direction). The poverty headcount ratio in particular is generally estimated well. As shown in Table 11, a marked difference is however observed when comparing the cases of China (a well-behaved distribution) and Brazil (a multi-peaked, heavily skewed distribution).[10] For lower poverty lines, the poverty headcount ratio is substantially more under-estimated in the case of Brazil, with biases as high as 5 to 14 percentage points. The two parameterizations give rise to concerning results for the poverty gap ratio and the squared poverty gap as well. Overall, the magnitude of the biases and the manner in which they change sign depending on the sample size, poverty line, and poverty indicator give rise to concern regarding the use of grouped data from underlying multi-peaked, highly unequal distributions.

As before, inequality is better estimated than poverty (Table 12). While our chosen household surveys have Gini coefficients ranging between 35 (Vietnam) and 71 (Brazil), we find that in all cases considered, the biases are negligible. For Brazil, the Beta functional form underestimates the Gini index by at most 2 percentage points. Generally, inequality is well estimated by both parameterizations.

Table 8
Poverty biases (in percentage points), Multimodal distribution

Poverty indicator	Poverty line Median x:	True	GQ Quintiles	GQ Deciles	GQ Ventiles	Beta Quintiles	Beta Deciles	Beta Ventiles
Poverty headcount ratio	1.33	71.3	1.95	-1.72	-1.79	0.38	-7.04	-9.74
	0.75	43.5	-2.55	-6.90	-7.83	-0.43	-1.05	-1.38
	$2/day	4.2	0.26	-	-	-0.60	-	-
	$1/day	0.0	-	-	-	-	-	-
Poverty gap ratio	1.33	34.90	0.34	-1.20	-1.40	0.65	-0.51	-0.44
	0.75	18.17	-0.92	0.26	0.32	-1.00	0.81	2.30
	$2/day	0.97	0.38	-	-	1.65	-	-
	$1/day	0.0	-	-	-	-	-	-
Squared poverty gap	1.33	21.33	0.41	1.47	1.15	0.61	1.00	1.97
	0.75	9.80	0.16	4.78	4.07	0.00	0.85	2.33
	$2/day	0.25	0.35	-	-	3.78	-	-
	$1/day	0.0	-	-	-	-	-	-

Note: The international $1.08/day poverty line (at 1993 international US$) produces zero poverty estimates since the lowest per capita GDP in 2004 was $515 (Sierra Leone) while the yearly equivalent of the $1.08/day poverty line is $448. Interestingly, this suggests that, for the $1/day international poverty line, the vast majority of global poverty can be associated (against the counterfactual of even national income distributions) with intra-national inequalities.

Table 9
Inequality biases (in percentage points)

Distribution:	True	GQ Quintiles	GQ Deciles	GQ Ventiles	Beta Quintiles	Beta Deciles	Beta Ventiles
Dagum	38.17	-0.22	-0.15	-0.12	0.03	-0.04	-0.04
Multimodal	52.43	5.10	3.59	4.53	3.73	0.64	-0.12

Discussion and conclusions

In this paper we have analyzed the biases associated with Lorenz curve, poverty, and inequality estimates obtained from grouped data using two parameterizations: Villasenor and Arnold (1989) and Kakwani (1980a). These estimation methods have been widely used through the World Bank's POVCAL and SimSIP computational tools. They have also been the basis for many regional and global assessments of poverty and inequality. We have used data drawn from a wide range of income distributions with single and multiple modes, and have undertaken both Monte Carlo simulations as well as deterministic comparisons.

We found that the two parameterizations perform relatively well in estimating

Table 10
Poverty headcount ratio biases (in percentage points)

Survey:	Poverty line:	True	GQ Quintiles	GQ Deciles	GQ Ventiles	Beta Quintiles	Beta Deciles	Beta Ventiles
Vietnam	$1/day	5.2	-1.00	-1.10	-0.19	-0.77	-0.51	n/a
	Capability	35.7	0.17	0.30	1.71	0.11	0.04	n/a
	$2/day	41.9	-0.40	-0.35	1.52	-0.35	-0.64	n/a
Tanzania	Capability	40.4	-0.30	-0.15	-0.13	-0.25	-0.23	-0.20
	$1/day	75.4	1.00	0.64	0.57	1.04	0.33	-0.23
	$2/day	97.8	0.34	0.19	0.16	n/a	0.96	0.94
Nicaragua	Capability	30.6	-0.45	-0.30	-0.24	-0.62	-0.42	-0.27
	$1/day	44.6	0.09	0.14	0.18	0.35	0.31	0.29
	$2/day	79.0	0.58	0.26	0.17	0.40	0.10	-0.12

poverty and inequality for unimodal distributions. Larger biases were identified, however, in the case of the multimodal distribution considered. The extent of misestimation of poverty does not vary predictably with the level of poverty line, the poverty indicator, or even the sample size. Inequality (measured by the Gini index) is usually well estimated.

We often found that the two parameterizations yielded negative fitted income shares, giving rise to invalid estimated Lorenz curves. Should one use or discard the poverty and inequality output associated with them? We found no straightforward correspondence between the validity of the Lorenz curve and the quality of the poverty and inequality estimates. Notably, Kakwani (1980b) defended a parameterization, which had been shown to give rise to theoretically invalid Lorenz curve estimates, arguing that its *overall* superior performance in fitting income distributions was a sufficient basis for its use.

The results presented in this study offer qualified support for the use of existing software, such as POVCAL and SimSIP, in poverty and inequality analysis based on grouped data. It should be noted, however, that other techniques have also been recently proposed to analyze such data. For example, Minoiu and Reddy (2007b) asssessed kernel density estimators and concluded that Lorenz curves functional forms often outperformed kernel methods in poverty analysis. Wu and Perloff (2003) evaluated the maximum entropy density estimator for grouped data and found that it performed well. Similarly, Chotikapanich, Griffiths, and Rao (2007) have shown the the Generalized Beta 2 distribution is a good parametric choice. Since grouped data will continue to be an important source of information - and often, the only one - in poverty and inequality analysis, future research should aim to establish more fully the relative empirical performance of these alternatives and

Table 11
Poverty biases (in percentage points). GQ estimation method.

	Quintiles		Deciles		Ventiles	
Poverty gap ratio:						
Poverty line: Median x	**China**	**Brazil**	**China**	**Brazil**	**China**	**Brazil**
2.00	-0.13	-0.2	0.05	-0.3	0.10	-0.4
1.50	0.17	0.0	0.30	-0.3	0.33	0.3
1.00	-0.55	6.5	-0.64	6.1	-0.67	7.7
0.50	0.94	-5.2	0.83	-5.2	0.79	-3.1
0.25	-2.43	-12.3	-2.04	-11.7	-1.94	-9.9
0.20	-6.30	-14.1	-5.71	-13.4	-5.56	-11.7
Poverty gap ratio:						
	Quintiles		**Deciles**		**Ventiles**	
Poverty line: Median x	**China**	**Brazil**	**China**	**Brazil**	**China**	**Brazil**
2.00	0.00	-2.5	0.04	-2.2	0.05	-1.0
1.50	-0.08	-3.2	-0.08	-2.7	-0.09	-1.3
1.00	-0.02	-4.0	-0.03	-3.1	-0.04	-1.4
0.50	-0.02	-8.8	0.03	-6.5	0.04	-5.3
0.25	-0.63	-8.7	-0.62	-4.5	-0.62	-4.0
0.20	0.52	-7.6	0.37	-2.5	0.33	-2.3
Squared poverty gap:						
	Quintiles		**Deciles**		**Ventiles**	
Poverty line: Median x	**China**	**Brazil**	**China**	**Brazil**	**China**	**Brazil**
2.00	-0.09	2.1	-0.08	-6.1	-0.08	-4.7
1.50	-0.14	-7.0	-0.13	-5.5	-0.14	-4.1
1.00	-0.20	-7.1	-0.18	-4.8	-0.17	-3.6
0.50	-0.52	-6.6	-0.47	-1.3	-0.46	-1.2
0.25	-0.94	2.8	-0.94	14.9	-0.94	12.5
0.20	-1.47	9.3	-1.43	25.6	-1.42	21.6

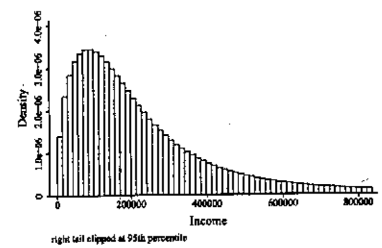
Table 12
Inequality biases (in percentage points)

		GQ			Beta		
Survey:	**True**	**Quintiles**	**Deciles**	**Ventiles**	**Quintiles**	**Deciles**	**Ventiles**
Brazil	71.04	-0.63	0.03	0.29	-2.00	-1.16	-0.58
China	38.6	-0.18	-0.11	-0.10	0.07	0.02	-0.04
Tanzania	37.2	0.58	0.56	-0.26	0.83	0.69	n/a
Vietnam	35.0	0.24	0.07	0.04	0.36	0.14	0.03
Nicaragua	45.2	0.22	0.08	0.09	0.19	0.09	0.08

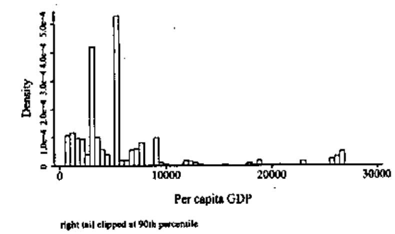
to develop new methods.

Appendix

Figure A1
Theoretical distributions used in Monte Carlo analysis



(a) Dagum distribution



(b) Multimodal (notional) distribution

Description of POVCAL and SimSIP

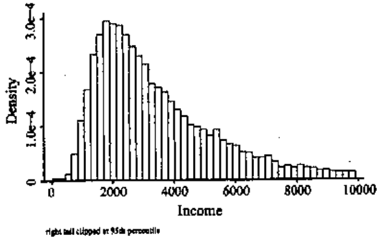
POVCAL (www.worldbank.org/lsms/tools/povcal) and SimSIP-Simulations for Social Indicators and Poverty (www.worldbank.org/simsip) are poverty assessment tools produced and distributed by the World Bank. POVCAL functions in MS-DOS, whereas SimSIP is Excel-based. SimSIP has the additional features that it enables sector-level and decomposition analyses of poverty.

The Lorenz curve is estimated from grouped data by regression analysis based on two functional forms - the GQ parameterization of Villasenor and Arnold (1989) and the Beta parameterization of Kakwani (1980a) - each involving three parameters (Chen, Datt, and Ravallion, 2001; and Datt, 1998). The grouped data read by the programs may take different forms (for example, income shares or mean incomes of population quantiles, share of the population in given income intervals, etc.). Datt (1998) and Ramadas, van der Mensbrugghe, and Wodon (2002) provide detailed formulas for poverty and inequality indices as functions of the parameters of the two functional forms.

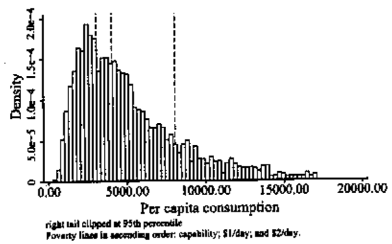
For a user-specified poverty line, the output includes: the poverty headcount ratio (FGT0), the poverty gap index (FGT1), the squared poverty gap (FGT2), and the elasticity of these poverty measures with respect to the mean income (assuming a constant distribution). Also reported are the Gini coefficient of inequality, the Lorenz curve, and the parameterization which provides a better fit to the data.

The programs also report on the consistency of the estimated Lorenz curve with the requirements for a valid Lorenz curve. It can easily be shown algebraically that the Beta functional form *always* violates conditions required for the validity of the Lorenz curve (in particular by implying a negative slope at the origin). The GQ

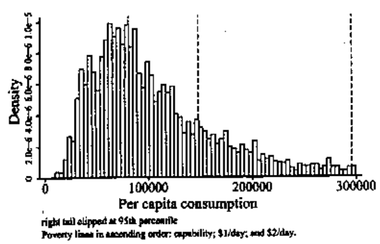
Figure A2
Household survey distributions used for deterministic comparisons



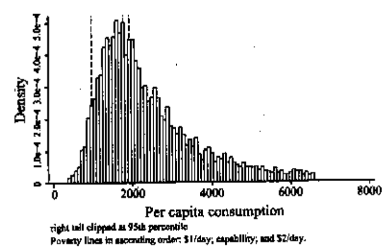
(a) China (1995)



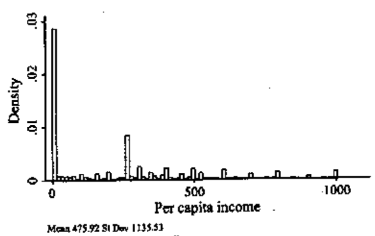
(b) Nicaragua (1998)



(c) Tanzania (2001)



(d) Vietnam (1998)



(e) Brazil (2004)

parameterization gives rise to valid Lorenz curves only under certain conditions on its parameters. Villasenor and Arnold (1989) find that the estimated GQ Lorenz curve is sometimes negative. We also find this to be the case, as the range of the regressors, and consequently the domain of the fitted values, are unrestricted in the estimation.

Description of income (consumption) variables in household surveys

The 2004 Brazilian National Household Sample Survey (Pesquisa Nacional por Amostra de Domicilios (PNAD)) is a representative household survey of the entire population of Brasil, with the exception of remote areas in the Amazon. The data are weighted. The income variable represents *individual total earned income*(labor income and pension transfers). The two modes represent a high mass at zero (reported income by groups, such as children and housewives) and the minimum wage (which is also the minimum value for pensions).

The 1995 Chinese Household Income Project is publicly available through the Inter-University Consortium for Political and Social Research, 2000. We pooled the rural and urban surveys to obtain the variable, which represents *per capita consumption*(with no adjustment for household composition). The data are not weighted. For a detailed description of this variable, see Reddy and Minoiu (2006).

The 1998 Vietnam Living Standards Survey (VLSS) contains information on *per capita expenditure* of households at current prices for 22,510 individuals. The data are weighted.[11]

The 1997-98 Nicaragua Living Standards and Measurement Survey (LSMS) contains information on *per capita consumption* for 18,383 individuals. The data are weighted.[12]

The 2000-01 Tanzania Household Budget Survey[13] contains information on *per capita consumption* for 22,176 households. The data are weighted.

Notes

1 *Acknowledgments:* We are grateful for the support of the Bureau of Development Policy and the International Poverty Centre of the United Nations Development Program. We would like to thank Terry McKinley for facilitating that support. We would also like to thank Shaohua Chen, Gaurav Datt, and Martin Ravallion for helpful comments, Sergey Kivalov for assisting us with software development, and Catherine Choi and Prabhjot Kaur for research assistance. Income data from the 2004 Brazilian National Household Sample Survey was generously provided by Marcelo Medeiros of the Instituto de Pesquisa Econômica Aplicada (IPEA). We are grateful to the editor and two anonymous referees for useful comments. The views expressed in this article are those of the authors and do not necessarily represent those of the International Monetary Fund (IMF) or IMF policy.

2 In what follows, we bear in mind that poverty and inequality analysis can be applied to distributions of consumption, income, wealth, or other dimensions of personal advantage. Without loss of generality, however, we refer here thoughout to distributions of income.

3 China's State Statistics Bureau has not made full household survey data available to outside researchers.

4 Some of the functions of POVCAL and SimSIP are built into the PovcalNet website of the World Bank (http://iresearch.worldbank.org/PovcalNet/jsp/index.jsp), which allows users to obtain poverty estimates for any country in the period covered for a specified poverty line. The two programs are briefly described in the Appendix.

5 Kakwani and Podder (1976) discuss the goodness of fit of their two-parameter specification. An empirical exercise which they undertake (using data from the 1967-68 Australian Survey of Consumer Expenditures and Finance) reveals underestimation of the mean income of the poorest 5 per cent of the population, and overestimation of the mean income of the poorest 10 per cent. Anand (1983), and Anand and Kanbur (1993a;b) note that there are reasons that Kakwani and Podder's (1976) proposed functional form for parametric estimation violates the theoretical requirements for a valid Lorenz Curve. Dhongde (2004) derived the small sample bias of Lorenz curve estimates associated with the earlier parameterization of Kakwani and Podder (1973).

6 The Dagum distribution has the following parameter values: $a = 2.742$, $b = 100{,}000$ and $c = 0.337$.

7 In the case of the Dagum distribution, we draw 100 random samples from a universe of 1,000,000 observations. For the multimodal distribution, we draw 100 random samples from a universe of almost 600,000 observations, each representing 1/10,000 of the world's population.

8 We often cannot report poverty biases for the $2/day poverty line due to the frequency with which POVCAL shut down, failed to write to the output files, provided meaningless output (e.g., higher than one poverty headcount

ratios) or generated infeasible bounds for the poverty lines. The biases are thus computed across successful program runs. For more examples on technical problems encountered when running POVCAL, see Minoiu and Reddy (2007a).

9 The findings presented in this section are consistent with those from analyzing theoretical distributions such as Weibull, Log-normal, Pareto, and Generalized Beta II. (The results are reported in Minoiu and Reddy, 2007a.)

10 We employ for Brazil a survey that includes (rather than discards) zero values for income, recognizing that this is only one plausible treatment of the underlying data.

11 Source: World Bank Living Standards Measurement Study (LSMS), Development Economics Research Group (DECRG), Washington, D.C.

12 Source: World Bank Living Standards Measurement Study, Development Economics Research Group (DECRG), Washington, D.C.

13 Source: National Bureau of Statistics, Tanzania, 2002.

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Maximum Entropy and the Entropy of Mixing for Income Distributions

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Over the last 100 years, a large number of distributions has been proposed for the modeling of size phenomena, notably the size distribution of personal incomes. The most widely known of these models are the Pareto, log-normal, generalized log-normal, Generalized Gamma, generalized Beta of the first and of the second kind, the Dagum, and the Singh-Madala distributions. They are discussed as a group in this note, as general forms of income distributions. Several well-known models are derived from them as sub-families with interesting applications in economics. The behaviour of their entropy is what is here under study. Maximum entropy formalism chooses certain forms of entropy and derives an exponential family of distributions under certain constraints. Finding constraints that income distributions have maximum entropy is another direction of this note. In economics and social statistics, the size distribution of income is the basis of concentration on the Lorenz curve. The difference between the tail of the Lorenz function and the Lorenz function itself determines the entropy of mixing. In the final section of this note, theoretical properties of well-known income distributions are also derived in view of the entropy of mixing.

Keywords: Maximum entropy, Lorenz curve, Lorenz order, income distributions, Generalized Beta distributions
JEL Classifications: D31, C02

Introduction

In June 1905, a paper written by Otto Lorenz truly revolutionized the economics and statistics of studying income distributions. Personal income distributions had

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